Information content versus word length in random typing

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Abstract. Recently, it has been claimed that a linear relationship between a measure of information content and word length is expected from word length optimization and it has been shown that this linearity is supported by a strong correlation between information content and word length in many languages (*Piantadosi et al. 2011, PNAS 108, 3825-3826*). Here, we study in detail some connections between this measure and standard information theory. The relationship between the measure and word length is studied for the popular random typing process where a text is constructed by pressing keys at random from a keyboard containing letters and a space behaving as a word delimiter. Although this random process does not optimize word lengths according to information content, it exhibits a linear relationship between information content and word length. The exact slope and intercept are presented for three major variants of the random typing process. A strong correlation between information content and word length can simply arise from the units making a word (e.g., letters) and not necessarily from the interplay between a word and its context as proposed by Piantadosi *et al.* In itself, the linear relation does not entail the results of any optimization process.

Keywords: Zipf's law of brevity, random typing, uniform information density.

PACS numbers: 89.70.-a Information and communication theory
89.75.Da Systems obeying scaling laws
05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion
02.30.Lt Sequences, series, and summability

Submitted to: *Journal of Statistical Mechanics: theory and experiment (JSTAT)*

1. Introduction

In his pioneering research, G. K. Zipf showed that more frequent words tend to be shorter [1], and parallels of this brevity law have been reported for the behavior of other species [2, 3]. Recently, it has been argued that "average information content is a much better predictor of word length than frequency" and that this "indicates that human lexicons are efficiently structured for communication by taking into account interword statistical dependencies." [4, p. 1]. According to the uniform information density hypothesis (e.g., [5]), "language users make choices that keep the number of bits of information communicated per unit of time approximately constant" and thus "the amount of information conveyed by a word should be linearly related to the amount of time it takes to produce –approximately, its length– to convey the same amount of information in each unit of time" [4, p. 1]. Here it will be shown that hitting keys from a keyboard at random (e.g., [6, 7]) generates words that reproduce this linear relationship. Therefore, the observation of such a linear relationship does not constitute unequivocal evidence for any kind of optimal choices made by speakers.

Throughout this paper, C denotes contexts and W denotes words. As in Ref. [4], the context of a word consists of a fixed number of preceding words, and the information content of a word w is given by

$$I(w) = -\sum_c p(C = c|W = w) \ln p(W = w|C = c).$$

The expected information content of words of length ℓ is defined as [4]

$$I(\ell) = \sum_{\|w\|=\ell} p(W = w|\|w\| = \ell) I(w), \quad (1)$$

where $\|w\|$ is the length (in letters) of a word w and ℓ is a fixed parameter value. In this study, we detail some connections between $I(w)$ and standard information theory measures. The definition of $I(w)$ that we borrow from Ref. [4] is somewhat idiosyncratic in relation to standard information-theory. We found that, Ref. [8], the reference supplied in Ref. [4] as a justification for Eq. 1, does not in fact justify the equation in any evident way. In this study we demonstrate that $I(\ell)$ is a linear function of ℓ for a general class of random typing processes. The only requirement is that the context is defined by means of neighbouring words (as in [4]) or that empty words (words of length zero) are allowed as in many variants of the random typing process [6, 9, 10].

2. Connections with standard information theory

We now introduce our basic notation and conventions. The self-information of an event that has probability p is $-\ln p$. We consider C and W independent if and only if $p(C = c, W = w) = p(C = c)p(W = w)$. As usual, by the definition of conditional probability, independence implies both $p(C = c|W = w) = p(C = c)$ and $p(W = w|C = c) = p(W = w)$, for any individual c and w. Therefore, under independence between C and W, it holds that $I(w) = I_0(w) = -\ln p(W = w)$, that is

to say, $I(w)$ is just the self-information of w. The expected self-information content of a word of length ℓ is

$$\begin{aligned} I_0(\ell) &= -\sum_{\|w\|=\ell} p(W=w|\|w\|=\ell)\ln p(W=w) \\ &= -\sum_{\|w\|=\ell} p(W=w|\|w\|=\ell)\ln p(W=w,\|w\|=\ell). \end{aligned} \quad (2)$$

In sum, under independence between C and W, $I(\ell)$ and $I_0(\ell)$ coincide.

The conditional entropy is defined as,

$$\begin{aligned} H(W|C) &= \sum_c p(C=c)H(W|C=c) \\ &= -\sum_c p(C=c)\sum_w p(W=w|C=c)\ln p(W=w|C=c). \end{aligned} \quad (3)$$

Given only the joint probability, i.e. $p(W=w,C=c)$, one can use Bayes' Theorem for calculating the conditional and marginal probabilities, as it was done in previous work [4] and is assumed by various information theoretic models of Zipf's law for word frequencies [11, 12]. Simple application of Bayes' Theorem to the definition of $H(W|C)$ in (3) shows that the conditional entropy is the expectation of $I(w)$:

$$\begin{aligned} H(W|C) &= -\sum_c\sum_w p(W=w,C=c)\ln p(W=w|C=c) \\ &= -\sum_w p(W=w)\sum_c \frac{p(W=w,C=c)}{p(W=w)}\ln p(W=w|C=c) \\ &= -\sum_w p(W=w)\sum_c p(C=c|W=w)\ln p(W=w|C=c) \\ &= \sum_w p(W=w)I(w) = E[I(w)]. \end{aligned} \quad (4)$$

It is not difficult to see that $I_0(w)$ is the upper bound of $I(w)$ and $H(C|w)$ is its lower bound; formally,

$$H(C|w) \leq I(w) \leq I_0(w). \quad (5)$$

As for a lower bound of $I(w)$, the relative entropy (or Kullback-Leibler divergence) between the context conditional probability and the word conditional probability is [13]

$$\begin{aligned} D(p(C=c|W=w)\|p(W=w|C=c)) &= \sum_c p(C=c|W=w)\ln\frac{p(C=c|W=w)}{p(W=w|C=c)} \\ &= \sum_c p(C=c|W=w)\ln p(C=c|W=w) \\ &\quad -\sum_c p(C=c|W=w)\ln p(W=w|C=c) \\ &= I(w) - H(C|w). \end{aligned}$$

Therefore $I(w) \geq H(C|w)$ by the non-negativity of the relative entropy [13]. As for the upper bound of $I(w)$, the non-negativity of mutual information, i.e. $I(W;C) = H(W) - H(W|C) \geq 0$ [13] and (4), yields

$$H(W|C) \leq H(W)$$

$$\sum_w p(W=w)I(w) \leq -\sum_w p(W=w)\ln p(W=w)$$
$$= \sum_w p(W=w)I_0(w)$$

if and only if $I(w) \leq I_0(w)$, as we wanted to prove. Combining (1) and (5) results in

$$I_C(\ell) \leq I(\ell) \leq I_0(\ell), \tag{6}$$

where $I_C(\ell)$ is defined as

$$I_C(\ell) = \sum_{\|w\|=\ell} p(W=w|\|w\|=\ell)H(C|w).$$

3. Information content versus length in random typing

Random typing [6, 10] is a process in which a sequence of characters is produced by sampling randomly from a set of possible characters. Here we consider a generalized random typing model based upon variants allowing for unequal letter probabilities as in [7, 10] and allowing one to specify a minimum word length [14].

Assume that characters are produced from an alphabet $\Sigma = \{\sigma_0, ..., \sigma_i, ..., \sigma_{\lambda-1}\}$, where λ is the alphabet size, σ_0 represents the word delimiter (i.e., the space character) and the remaining characters of Σ are letters. We assume that all the characters in Σ are produced at random and independently, with the only exception that two instances of the space character must be separated by at least ℓ_0 intervening characters other than the space. In such model, the production of a word is separated into two phases: generation of the space-free prefix of length ℓ_0, and generation of the remainder. S is a random variable taking values from Σ as generated by the random typing process. $p_\Sigma(S=s)$ is defined as the probability of producing character s as the k-*th* character after the last space produced (or after the beginning of the sequence if no space has been produced yet), for any value $k \geq \ell_0$. $p_{\Sigma\setminus\{\sigma_0\}}(S=s)$ is the same probability as $p_\Sigma(S=s)$ for values of $k < \ell_0$. The abbreviation $p_0 = p_\Sigma(S=\sigma_0)$ will be used hereafter. We assume that $p_\Sigma(S=s) > 0$ for all characters in Σ with the additional constraint that $p_0 < 1$. $p_{\Sigma\setminus\{\sigma_0\}}(S=s)$ is defined in terms of $p_\Sigma(S=s)$,

$$p_{\Sigma\setminus\{\sigma_0\}}(S=s) = \begin{cases} \frac{p_\Sigma(S=s)}{1-p_0} & \text{if } s \neq \sigma_0 \\ 0 & \text{if } s = \sigma_0. \end{cases}$$

The generalized random typing process with unequal letter probabilities is defined by λ parameters: ℓ_0 and the $\lambda - 1$ probabilities $p_\Sigma(S=\sigma_i)$ for $0 \leq i \leq \lambda - 2$ with

$$p_\Sigma(S=\sigma_{\lambda-1}) = 1 - \sum_{i=0}^{\lambda-2} p_\Sigma(S=\sigma_i).$$

Notice the additional parameter ℓ_0 that is not considered in other versions of the random typing model and allows for unequal character probabilities [7, 10].

In the remainder of this section we start by proving that $I_0(\ell)$ is a linear function of ℓ, providing exact analytical expressions for its slope and intercept. We continue by showing that $I(\ell)$ can be inferred from $I_0(\ell)$. If the context is defined by words, as in

Ref. [4], then $I(\ell) = I_0(\ell)$ because our generalized random typing process produces words independently from the previous ones. If the context are characters, then $I(\ell) = I_0(\ell)$ is also warranted when $\ell_0 = 0$ because this is the case where self-repulsion of the space is suppressed. When $\ell_0 > 0$, (6) indicates that $I(\ell)$ cannot exceed $I_0(\ell)$.

In order to calculate the probability of producing a concrete word $w = s_1, ..., s_i, ..., s_\ell$, where s_i is the i-th character from Σ of w, we use the shorthand

$$\mathcal{P}_{i,j} = \prod_{h=i}^{j} p_\Sigma(S = s_h).$$

By the independence between characters (except for space self-repulsion at distances smaller than ℓ_0), the probability that a random word W that has length ℓ coincides with $w = s_1, ..., s_i, ..., s_\ell$ is

$$\begin{aligned} p(W = w, \|w\| = \ell) &= \left(\prod_{i=1}^{\ell_0} p_{\Sigma\backslash\{\sigma_0\}}(S = s_i)\right)\left(\prod_{i=\ell_0+1}^{\ell} p_\Sigma(S = s_i)\right) p_0 \\ &= \frac{p_0}{(1-p_0)^{\ell_0}}\left(\prod_{i=1}^{\ell} p_\Sigma(S = s_i)\right) \\ &= \frac{p_0}{(1-p_0)^{\ell_0}}\mathcal{P}_{1,\ell}, \end{aligned} \tag{7}$$

the probability that a word has length ℓ is

$$p(\|w\| = \ell) = p_0(1-p_0)^{\ell-\ell_0}$$

and the probability of a word w given its length is therefore

$$\begin{aligned} p(W = w | \|w\| = \ell) &= \frac{p(W = w, \|w\| = \ell)}{p(\|w\| = \ell)} \\ &= \frac{1}{(1-p_0)^{\ell}}\mathcal{P}_{1,\ell}. \end{aligned} \tag{8}$$

Applying (7), the self-information of a word w of length ℓ is

$$-\ln p(W = w, \|w\| = \ell) = b - \sum_{i=1}^{\ell} \ln p_\Sigma(S = s_i), \tag{9}$$

where b is defined as

$$b = \ln \frac{(1-p_0)^{\ell_0}}{p_0}. \tag{10}$$

Combining (8) and (9) with the definition of $I_0(\ell)$ in (2), gives

$$I_0(\ell) = \frac{1}{(1-p_0)^{\ell}} \sum_{s_1,\dots,s_\ell} \mathcal{P}_{1,\ell}\left(b - \sum_{i=1}^{\ell} \ln p_\Sigma(S = s_i)\right).$$

Bearing in mind that

$$\begin{aligned} \sum_{s_1,\dots,s_\ell} \mathcal{P}_{1,\ell} &= \sum_{s_1\in\Sigma\backslash\{\sigma_0\}} \cdots \sum_{s_i\in\Sigma\backslash\{\sigma_0\}} \cdots \sum_{s_\ell\in\Sigma\backslash\{\sigma_0\}} \mathcal{P}_{1,\ell} \\ &= \sum_{s_1\in\Sigma\backslash\{\sigma_0\}} \cdots \sum_{s_i\in\Sigma\backslash\{\sigma_0\}} \cdots \sum_{s_\ell\in\Sigma\backslash\{\sigma_0\}} \prod_{h=1}^{\ell} p_\Sigma(S = s_h) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{s_1 \in \Sigma \setminus \{\sigma_0\}} \dots \sum_{s_i \in \Sigma \setminus \{\sigma_0\}} \dots \mathcal{P}_{1,\ell-1} \sum_{s_\ell \in \Sigma \setminus \{\sigma_0\}} p_\Sigma(S = s_\ell) \\
&= (1-p_0) \sum_{s_1 \in \Sigma \setminus \{\sigma_0\}} \dots \sum_{s_i \in \Sigma \setminus \{\sigma_0\}} \dots \sum_{s_{\ell-1} \in \Sigma \setminus \{\sigma_0\}} \mathcal{P}_{1,\ell-1} \\
&= (1-p_0)^2 \sum_{s_1 \in \Sigma \setminus \{\sigma_0\}} \dots \sum_{s_i \in \Sigma \setminus \{\sigma_0\}} \dots \sum_{s_{\ell-2} \in \Sigma \setminus \{\sigma_0\}} \mathcal{P}_{1,\ell-2} \\
&= \dots \\
&= (1-p_0)^\ell,
\end{aligned}
$$

one can write

$$
I_0(\ell) = b + \frac{1}{(1-p_0)^\ell} \sum_{s_1,\dots,s_\ell} \mathcal{P}_{1,\ell} \left(-\sum_{i=1}^{\ell} \ln p_\Sigma(S = s_i) \right). \tag{11}
$$

Notice that

$$
\begin{aligned}
&\sum_{s_1,\dots,s_\ell} \mathcal{P}_{1,\ell}(-\ln p_\Sigma(S = s_i) &= \\
&\sum_{s_1,\dots,s_{j-1},s_{j+1},\dots,s_\ell} \left[\mathcal{P}_{1,j-1}\mathcal{P}_{j+1,\ell} \sum_{s_j \in \Sigma \setminus \{\sigma_0\}} -p_\Sigma(S = s_j) \ln p_\Sigma(S = s_j) \right] &= \\
&(H_\Sigma(S) + p_0 \ln p_0) \sum_{s_1,\dots,s_{j-1},s_{j+1},\dots,s_\ell} \mathcal{P}_{1,j-1}\mathcal{P}_{j+1,\ell} &= \\
&(H_\Sigma(S) + p_0 \ln p_0)(1-p_0)^{\ell-1},
\end{aligned} \tag{12}
$$

where

$$
\begin{aligned}
H_\Sigma(S) &= -\sum_{s \in \Sigma} p_\Sigma(S = s) \ln p_\Sigma(S = s) \\
&= -\sum_{s \in \Sigma \setminus \{\sigma_0\}} p_\Sigma(S = s) \ln p_\Sigma(S = s) - p_0 \ln p_0
\end{aligned} \tag{13}
$$

is the character entropy after the space-free prefix of length ℓ_0. Therefore, applying (12) to (11) one finally obtains $I_0(\ell) = a\ell + b$, where

$$
a = \frac{1}{1-p_0}(H_\Sigma(S) + p_0 \ln p_0)
$$

and b is defined as in (10). Notice that the slope a is always positive because $H_\Sigma(S) \geq 0$ as any entropy and, according to (13), $H_\Sigma(S) > p_0 \ln p_0$ provided that $\lambda > 1$ (recall that no character from Σ has probability zero of occurring after the free-space prefix). Therefore, $I_0(\ell)$ grows linearly with ℓ for $\lambda > 1$.

Table 1 summarizes the parameters of the linear relationship between $I_0(\ell)$ for our generalized random typing process and two particular cases: (a) equal letter probabilities (all characters except the space must be equally likely) [14] and (b) equal character probabilities (all characters including the space are equally likely) and empty words are allowed, i.e. $\ell_0 = 0$ [9]. Notice that (b) is a particular case of (a). Variant (a) [14] means that

$$
p_\Sigma(S = s) = \begin{cases} \frac{1-p_0}{\lambda-1} & \text{if } s \neq \sigma_0 \\ p_0 & \text{if } s = \sigma_0, \end{cases}
$$

Table 1. Summary of the linear dependency between the self-information content as a function of word length, $I_0(\ell) = a + b$, and related quantities for three major variants of the random typing process. $H_\Sigma(S)$ is the entropy of characters after the free-space prefix of length ℓ_0, p_0 is the probability of space and λ is the cardinality of Σ. p_s is used as a shorthand for $p_\Sigma(S = s)$.

	Random typing		
	Generalized	Equal letter probabilities [14]	Equal character probabilities (with $\ell_0 = 0$ [9])
a	$\frac{1}{1-p_0}(H_\Sigma(S) + p_0 \ln p_0)$	$\ln \frac{\lambda-1}{1-p_0}$	$\ln \lambda$
b	$\ln \frac{(1-p_0)^{\ell_0}}{p_0}$	$\ln \frac{(1-p_0)^{\ell_0}}{p_0}$	$\ln \lambda$
$H_\Sigma(S)$	$-\sum_{s \in \Sigma \setminus \{\sigma_0\}} p_s \ln p_s$ $-p_0 \ln p_0$	$(1-p_0) \ln \frac{\lambda-1}{1-p_0}$ $-p_0 \ln p_0$	$\ln \lambda$
p_0	p_0	p_0	$\frac{1}{\lambda}$
$p(W = w, \|w\| = \ell)$	$\frac{p_0}{(1-p_0)^{\ell_0}} \mathcal{P}_{1,\ell}$	$\frac{(1-p_0)^{(\ell-\ell_0)} p_0}{(\lambda-1)^\ell}$	$\frac{1}{\lambda}$
$p(W = w \| \|w\| = \ell)$	$\frac{1}{(1-p_0)^\ell} \mathcal{P}_{1,\ell}$	$\frac{1}{(\lambda-1)^\ell}$	$\frac{1}{(\lambda-1)^\ell}$

and is defined only by three parameters: ℓ_0, λ and p_0. The random typing process defined in [6] is a particular case with $\ell_0 = 0$. In a random typing process with equal letter probabilities, the character entropy after the space-free prefix is

$$
\begin{aligned}
H_\Sigma(S) &= (\lambda - 1)\left(-\frac{1-p_0}{\lambda-1} \ln \frac{1-p_0}{\lambda-1}\right) - p_0 \ln p_0 \\
&= (1-p_0) \ln \frac{\lambda-1}{1-p_0} - p_0 \ln p_0.
\end{aligned}
$$

Variant (b), the simplest random typing that has ever been presented to our knowledge, is defined with only one parameter, i.e. λ ($\ell_0 = 0$ and $p_0 = 1/\lambda$ in that case). (b) is known as the fair die rolling experiment [9] (see [7] for a version with $\ell_0 = 1$ and $p_0 = 1/\lambda$).

4. Conclusion

We have shown that $I(\ell) = a\ell + b$ does not imply that speakers have made optimal choices as argued in [4]. Uniform information density or related hypotheses (e.g., [5]) are not at all necesary to account for the linear correlation between $I(\ell)$ and ℓ: typing at random yields the same dependency independently from context. Our main point is that a linear correlation between information content and word length may simply arise internally, from the units making a word (e.g., letters) and not necessarily from the interplay between words and their context as suggested in [4]. However, future research should investigate if the parameters of the linear relationship predicted by random typing coincide with those of real texts or if a linear relationship is sufficient to account for the actual dependency between $I(\ell)$ and ℓ in real languages as it is suggested by the long-range correlations in texts at the level of words [15] or letters [16, 17] and

the differences between random typing and real language at the level of the distribution of word frequencies [14, 18] or word lengths [19].

Acknowledgments

This work was supported by the project OpenMT-2 (TIN2009-14675-C03) from the Spanish Ministy of Science and Innovation (RFC).

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